

SUBMANIFOLDS WITH NONPOSITIVE EXTRINSIC CURVATURE

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ABSTRACT. We prove that complete submanifolds, on which the Omori-Yau weak maximum principle for the Hessian holds, with low codimension and bounded by cylinders of small radius must have points rich in large positive extrinsic curvature. The lower the codimension is, the richer such points are. The smaller the radius is, the larger such curvatures are. This work unifies and generalizes several previous results on submanifolds with nonpositive extrinsic curvature.

1. INTRODUCTION

The results of this article show that isometric immersions $f : M^m \rightarrow \tilde{M}^n$ with low codimension and nonpositive extrinsic curvature at any point must satisfy strong geometric conditions. The simplest result along this line is that a two-dimensional surface with nonpositive curvature in \mathbb{R}^3 cannot be compact. This is a consequence of the well-known fact that at a point of maximum of a distance function on a compact surface in \mathbb{R}^3 the Gaussian curvature must be positive. It turns out that the simple idea in the proof of this elementary fact has far-reaching generalizations for non necessarily compact submanifolds in fairly general ambient Riemannian manifolds.

One of the main tools to extend this idea to higher dimensions and codimensions is an algebraic lemma due to Otsuki [13], which will be stated in next section. On the other hand, a key ingredient to handle the noncompact case is a maximum principle due to Omori [11] and generalized by Pigola-Rigoli-Setti [14]. Using this principle, Alías-Bessa-Dajczer [2] obtained an estimate for the mean curvature of an isometric immersion $f : M^m \rightarrow N^{n+l} = P^n \times \mathbb{R}^l$, under some assumptions on the manifold P^n , whose projection onto the first factor is bounded, the so-called cylindrically bounded submanifolds. More recently, Alías-Bessa-Montenegro [3] have provided an estimate for the extrinsic curvatures of such submanifolds.

In the statement below and the sequel, ρ stands for the distance function to a given reference point in M^m , $\log^{(j)}$ is the j -th iterate of the logarithm and $t \gg 1$ means that t is sufficiently large. Also $B_P[R]$ denotes the closed geodesic ball with radius R centered at a point o of P^n and $\text{inj}_P(o)$ is the injectivity radius of P^n at o . Finally, $K_M(\sigma)$ denotes the sectional curvature of M^m at a point $x \in M^m$

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along the plane $\sigma \subset T_x M$, and similarly for N^{n+l} , $K_f(\sigma) := K_M(\sigma) - K_N(f_*\sigma)$ is the *extrinsic sectional curvature* of f at x along σ and K_P^{rad} is the radial sectional curvatures of P^n with respect to o , that is, the sectional curvatures of tangent planes to P^n containing the vector $\text{grad}^P r$, where r is the distance function to o in P^n .

Theorem 1 (Alías-Bessa-Montenegro, [3]). *Let $f : M^m \rightarrow N^{n+l} = P^n \times \mathbb{R}^l$ be an isometric immersion with codimension $p = n + l - m < m - l$ of a complete Riemannian manifold whose scalar curvature satisfies*

$$(1.1) \quad s_M(x) \geq -A^2 \rho^2(x) \prod_{j=1}^J \left(\log^{(j)}(\rho(x)) \right)^2, \quad \rho(x) \gg 1,$$

for some constant $A > 0$ and some integer $J \geq 1$. Assume that $f(M) \subset B_P[R] \times \mathbb{R}^l$, with $0 < R < \min \left\{ \text{inj}_P(o), \frac{\pi}{2\sqrt{b}} \right\}$, where $\frac{\pi}{2\sqrt{b}}$ is replaced by $+\infty$ if $b \leq 0$. If $K_P^{\text{rad}} \leq b$ in $B_P[R]$, then

$$(1.2) \quad \sup_M K_f \geq C_b^2(R),$$

where

$$C_b(t) = \begin{cases} \sqrt{b} \cot(\sqrt{b}t) & \text{if } b > 0 \text{ and } 0 < t < \frac{\pi}{2\sqrt{b}}, \\ \frac{1}{t} & \text{if } b = 0 \text{ and } t > 0, \\ \sqrt{-b} \coth(\sqrt{-b}t) & \text{if } b < 0 \text{ and } t > 0. \end{cases}$$

Moreover,

$$(1.3) \quad \sup_M K_M \geq C_b^2(R) + \inf_{B_P[R]} K_P.$$

Remark 1. The geometry of the Euclidean factor \mathbb{R}^l plays essentially no role in the proof of the above result. Indeed, estimate (1.3) remains true if the former is replaced by any Riemannian manifold Q^l , which need not be even complete, whereas for (1.2) the only requirement is that K_Q be bounded from above (see comment below Theorem 2).

It is worth pointing out that the codimension restriction $p < m - l$ cannot be relaxed. Actually, it implies that $n > 2$ and $m > l + 1$. In particular, in a three-dimensional ambient space N^3 , that is, $n + l = 3$, we have that $l = 0$, and therefore $f(M) \subset B_P[R]$. In fact, the flat cylinder $\mathbb{S}^1(R) \times \mathbb{R} \subset B_{\mathbb{R}^2}[R] \times \mathbb{R}$ shows that the restriction $p < m - l$ is necessary.

On the other hand, estimates (1.2) and (1.3) are sharp. Indeed, the function C_b is well-known: the geodesic sphere $\partial B_{\mathbb{Q}_b^m}(R)$ of radius R in the simply connected complete space form \mathbb{Q}_b^m of constant sectional curvature b , with $R < \frac{\pi}{2\sqrt{b}}$ if $b > 0$, is an umbilical hypersurface with principal curvatures being precisely $C_b(R)$. It shows that its extrinsic and intrinsic sectional curvatures are constant and equal to $C_b^2(R)$ and $C_b^2(R) + b$, respectively, the latter following from the former by the Gauss equation. Then, for every $n > 2$ and $l \geq 0$ we can consider $M^{m-1+l} = \partial B_{\mathbb{Q}_b^m}(R) \times \mathbb{R}^l$ and take $f : M^{m-1+l} \rightarrow B_{\mathbb{Q}_b^m}[R] \times \mathbb{R}^l$ to be the canonical isometric embedding. Therefore $\sup_M K_f$ and $\sup_M K_M$ are the constant extrinsic and intrinsic sectional curvatures $C_b^2(R)$ and $C_b^2(R) + b$ of $\partial B_{\mathbb{Q}_b^m}(R)$, respectively.

The purpose of this paper is to provide a more accurate conclusion than the one of Theorem 1 by precisising how much extrinsic (respectively, intrinsic) sectional curvature satisfying estimate (1.2) (respectively (1.3)) appears depending on how low the codimension is. The idea is that the lower the codimension is, the more extrinsic (respectively, intrinsic) sectional curvature satisfying (1.2) (respectively (1.3)) will appear. Our main result can be stated as follows.

Theorem 2. *Let $f : M^m \rightarrow N^{n+l} = P^n \times Q^l$ be an isometric immersion with codimension $p = n + l - m < m - l$ of a complete Riemannian manifold whose radial sectional curvatures satisfy*

$$(1.4) \quad K_M^{\text{rad}}(x) \geq -A^2 \rho^2(x) \prod_{j=1}^J \left(\log^{(j)}(\rho(x)) \right)^2, \quad \rho(x) \gg 1,$$

for some constant $A > 0$ and some integer $J \geq 1$. Assume that $f(M) \subset B_P[R] \times Q^l$, with $0 < R < \min \left\{ \text{inj}_P(o), \frac{\pi}{2\sqrt{b}} \right\}$, where $\frac{\pi}{2\sqrt{b}}$ is replaced by $+\infty$ if $b \leq 0$. If $K_P^{\text{rad}} \leq b$ in $B_P[R]$, then

$$(1.5) \quad \sup_M \min \left\{ \max_{\sigma \subset W} K_f(\sigma) : \dim W > p + l \right\} \geq C_b^2(R).$$

Moreover,

$$(1.6) \quad \sup_M \min \left\{ \max_{\sigma \subset W} K_M(\sigma) : \dim W > p + l \right\} \geq C_b^2(R) + \inf_{B_P[R]} K_P.$$

The estimates of Theorem 2 are clearly better than the ones of Theorem 1. Actually, (1.5) and (1.6) reduce to (1.2) and (1.3), respectively, only in the case of the highest allowed codimension $p = m - 1 - l$. On the other hand, although we make a stronger assumption on the curvature of M^m , if (1.1) holds but (1.4) does not, then, since the scalar curvature is an average of sectional curvatures, we have that $\sup_M K_M = +\infty$, and hence (1.3) is trivially satisfied. Moreover, K_P is clearly bounded in $B_P[R]$, thus if also K_Q is bounded from above, we conclude that $\sup_M K_f = +\infty$ by the Gauss equation, so that (1.2) also holds trivially in this case. Finally, note that the same example considered below Theorem 1 shows that our estimates (1.5) and (1.6) are also sharp.

Remark 2. Theorem 2 is a special case of the much stronger result, Theorem 12, given in Section 3.

Let $f : M^m \rightarrow N^{n+l} = P^n \times Q^l$ be an isometric immersion. Recall that f is said to be *cylindrically bounded* if there exists a (closed) geodesic ball $B_P[R]$ of P^n , centered at a point $o \in P^n$ with radius $R > 0$, such that

$$(1.7) \quad f(M) \subset B_P[R] \times Q^l.$$

Otherwise, we say that f is *cylindrically unbounded*. Denote by R_f the *extrinsic radius* of a cylindrically bounded isometric immersion f (from o), that is, the smallest R for which (1.7) holds. As a consequence of Theorem 2, we have the following versions of the extrinsic radius results of Alías-Bessa-Montenegro [3].

Corollary 1. *Let $f : M^m \rightarrow N^{n+l} = P^n \times Q^l$ be an isometric immersion with codimension $p = n + l - m < m - l$ of a complete Riemannian manifold whose radial*

sectional curvatures satisfy (1.4). Assume that P^n is a complete Riemannian manifold with a pole and radial sectional curvatures $K_P^{\text{rad}} \leq b \leq 0$. If f is cylindrically bounded, then

$$\sup_M \min \left\{ \max_{\sigma \subset W} K_f(\sigma) : \dim W > p + l \right\} > -b$$

and the extrinsic radius satisfies

$$(1.8) \quad R_f \geq C_b^{-1} \left(\sqrt{\sup_M \min \left\{ \max_{\sigma \subset W} K_f(\sigma) : \dim W > p + l \right\}} \right).$$

In particular, if

$$\sup_M \min \left\{ \max_{\sigma \subset W} K_f(\sigma) : \dim W > p + l \right\} \leq -b,$$

then f is cylindrically unbounded.

Corollary 2. Let $f : M^m \rightarrow N^{n+l} = \mathbb{S}^n \times Q^l$ be an isometric immersion with codimension $p = n + l - m < m - l$ of a complete Riemannian manifold whose radial sectional curvatures satisfy (1.4). If

$$\sup_M \min \left\{ \max_{\sigma \subset W} K_M(\sigma) : \dim W > p + l \right\} \leq 1,$$

then

$$(1.9) \quad R_f \geq \frac{\pi}{2}.$$

On the other hand, a sharp lower bound for the Ricci curvature of bounded complete Euclidean hypersurfaces was obtained by Leung [8] and extended by Veeravalli [21] to nonflat ambient space forms.

Theorem 3 (Veeravalli, [21]). Let $f : M^n \rightarrow \mathbb{Q}_b^{n+1}$ be a complete hypersurface with sectional curvature bounded away from $-\infty$ such that $f(M) \subset B_{\mathbb{Q}_b^{n+1}}[R]$, with $R < \frac{\pi}{2\sqrt{b}}$ if $b > 0$. Then

$$(1.10) \quad \sup_M \text{Ric}_M \geq C_b^2(R) + b,$$

where Ric_M is the Ricci curvature of M^n .

Theorem 2 also gives an improvement of the above result, where we consider hypersurfaces of much more general ambient spaces and obtain that estimate (1.10) actually holds for the scalar curvature. This shows the unifying character of our result.

Corollary 3. Let $f : M^n \rightarrow P^{n+1}$ be a complete hypersurface whose radial sectional curvatures satisfy (1.4). Assume that $f(M) \subset B_P[R]$, with R as in Theorem 2. If $K_P^{\text{rad}} \leq b$ in $B_P[R]$, then

$$\sup_M s_M \geq C_b^2(R) + \inf_{B_P[R]} K_P.$$

Again observe that for the geodesic sphere $M^n = \partial_{\mathbb{Q}_b^{n+1}}(R)$ of radius R in \mathbb{Q}_b^{n+1} the above inequality is in fact an equality. Corollary 3 leads to similar extrinsic radius results to Corollaries 1 and 2 and, in particular, a criterion of unboundness:

Corollary 4. *Let $f : M^n \rightarrow P^{n+1}$ be a complete hypersurface whose radial sectional curvatures satisfy (1.4). Assume that P^{n+1} is a complete Riemannian manifold with a pole and sectional curvatures $K_P \geq c$ and $K_P^{\text{rad}} \leq b \leq 0$. If $f(M)$ is bounded, then $\sup_M s_M > c - b$ and*

$$(1.11) \quad R_f \geq C_b^{-1} \left(\sqrt{\sup_M s_M - c} \right).$$

In particular, if $\sup_M s_M \leq c - b$, then $f(M)$ is unbounded.

Corollary 5. *Let $f : M^n \rightarrow \mathbb{S}^{n+1}$ be a complete hypersurface whose radial sectional curvatures satisfy (1.4). If $\sup_M s_M \leq 1$, then*

$$(1.12) \quad R_f \geq \frac{\pi}{2}.$$

Finally, we also generalize in the same spirit of Theorem 2 the second part of the work of Alías-Bessa-Montenegro [3], concerning proper complete cylindrically bounded submanifolds with the norm of the second fundamental form with certain controlled growth.

Theorem 4. *Let $f : M^m \rightarrow N^{n+l} = P^n \times Q^l$ be a proper isometric immersion with codimension $p = n + l - m < m - l$ of a complete Riemannian manifold. Assume that $f(M) \subset B_P[R] \times Q^l$, with R as in Theorem 2. If $K_P^{\text{rad}} \leq b$ in $B_P[R]$, Q^l is a complete Riemannian manifold with a pole and*

$$(1.13) \quad \sup_{f^{-1}(B_P[R] \times \partial B_Q(t))} \|\alpha\| \leq \varsigma(t),$$

where α is the second fundamental form of f and $\varsigma : [0, +\infty) \rightarrow \mathbb{R}$ is a positive function satisfying $\int_0^{+\infty} 1/\varsigma = +\infty$, then (1.5) and (1.6) hold.

For hypersurfaces, the growth rate of the norm of the second fundamental form can be improved as follows.

Theorem 5. *Let $f : M^n \rightarrow N^{n+1} = P^{n+1-l} \times Q^l$, $n-l > 1$, be a properly immersed complete hypersurface with $f(M) \subset B_P[R] \times Q^l$, with R as in Theorem 2. Suppose that N^{n+1} satisfies the assumptions as in Theorem 4 and the second fundamental form α satisfies*

$$(1.14) \quad \sup_{f^{-1}(B_P[R] \times \partial B_Q(t))} \|\alpha\| \leq \varsigma^2(t),$$

where $\varsigma : [0, +\infty) \rightarrow \mathbb{R}$ is a positive function satisfying

$$\int_0^{+\infty} \frac{1}{\varsigma} = +\infty \text{ and } \limsup_{t \rightarrow +\infty} \frac{1}{\varsigma(t)} < +\infty.$$

Then (1.5) and (1.6) hold.

2. PRELIMINARIES

Our main tools to build the proof of Theorem 2 are Otsuki's Lemma, the Omori-Yau maximum principle and the Hessian comparison theorem, which for the sake of organization will be presented in two subsections.

2.1. Otsuki's Lemma. Throughout this subsection, V^n and W^p will be real vector spaces of dimensions n and p , respectively, endowed with positive definite inner products. For a symmetric bilinear form $\alpha : V^n \times V^n \rightarrow W^p$, we denote

$$K_\alpha(X, Y) = \langle \alpha(X, X), \alpha(Y, Y) \rangle - \|\alpha(X, Y)\|^2,$$

for any pair of vectors $X, Y \in V^n$. If σ is a two-dimensional subspace of V^n , we define

$$K_\alpha(\sigma) = \frac{K_\alpha(X, Y)}{\|X \wedge Y\|^2},$$

where $\{X, Y\}$ is any basis of σ and $\|X \wedge Y\|^2 = \|X\|^2 \|Y\|^2 - \langle X, Y \rangle^2$. Given an isometric immersion $f : M^m \rightarrow \tilde{M}^n$ with second fundamental form α , then for any $x \in M^m$ and any plane $\sigma \subset T_x M$ the Gauss equation yields

$$K_\alpha(\sigma) = K_f(\sigma).$$

A basic tool in this article is the following algebraic lemma, known as Otsuki's Lemma (for a proof see, for instance, [7]).

Lemma 1. *Let $\alpha : V^n \times V^n \rightarrow W^p$ be a symmetric bilinear form. Suppose there exists a real number $\lambda \geq 0$ such that*

- (i) $K_\alpha(\sigma) \leq \lambda$ for every plane $\sigma \subset V^n$,
- (ii) $\|\alpha(X, X)\| > \sqrt{\lambda}$ for every unit vector $X \in V^n$.

Then $p \geq n$.

Given a symmetric bilinear form $\alpha : V^n \times V^n \rightarrow W^p$, a vector $X \in V^n$ is said to be an *asymptotic vector* of α if

$$\alpha(X, X) = 0.$$

In the next statement and the sequel, we write $K_\alpha \leq 0$ (respectively, $K_\alpha < 0$) as a shorthand for $K_\alpha(\sigma) \leq 0$ (respectively, $K_\alpha(\sigma) < 0$) for any plane $\sigma \subset V^n$.

Corollary 6. *Let $\alpha : V^n \times V^n \rightarrow W^p$ be a symmetric bilinear form.*

- (i) *If $K_\alpha \leq 0$, then any subspace $S \subset V^n$, with $\dim S > p$, contains a nonzero asymptotic vector of α .*
- (ii) *If $K_\alpha < 0$, then $p \geq n - 1$.*

Proof. (i) This is just an equivalent way of stating Lemma 1 for $\lambda = 0$.

(ii) If there are no nonzero asymptotic vectors of α , the result follows from Lemma 1. Suppose $p < n - 1$, and assume that there exists a nonzero vector $X_0 \in V^n$ such that $\alpha(X_0, X_0) = 0$. Denote by U the orthogonal complement to X_0 in V^n , and consider the linear map $B_{X_0} : U \rightarrow W^p$ defined by $B_{X_0}(Y) = \alpha(X_0, Y)$. Since $\dim U = n - 1 > p$, there exists a nonzero vector $Y_0 \in U$ such that $B_{X_0}(Y_0) = 0$. This fact, together with $\alpha(X_0, X_0) = 0$, contradicts the assumption. \square

The following result is a direct consequence of Corollary 6-(ii).

Theorem 6. *Let $f : M^n \rightarrow \tilde{M}^{n+p}$ be an isometric immersion. Assume that there exists a point $x_0 \in M^n$ and a subspace $V_{x_0} \subset T_{x_0} M$ with dimension d such that $K_f(\sigma) < 0$ along every plane $\sigma \subset V_{x_0}$. Then $p \geq d - 1$.*

The preceding inequality is sharp, as shown, e.g., by the n -dimensional Clifford torus in \mathbb{S}^{2n-1} . Theorem 6 comes from a purely algebraic restriction on the “codimension” p of symmetric bilinear forms $\alpha : V^n \times V^n \rightarrow W^p$ with $K_\alpha < 0$, which gives its punctual nature. If, on the other hand, $K_f(\sigma) \leq 0$ in the above statement, it is possible to use part (i) of Corollary 6 to obtain the stronger restriction $p \geq d$, provided that the subspace V_{x_0} is free of asymptotic directions. Actually, this is a central idea in the study of the global implications of nonpositive extrinsic curvature in low codimension. The presence of a certain amount of nonpositive extrinsic sectional curvature everywhere together with some global assumption that can guarantee the existence of points without asymptotic directions must imply codimension restrictions. For compact Riemannian manifolds, for instance, one obtains the following generalization of a result for the flat case due to Tompkins [20].

Theorem 7. *Let M^n be a compact Riemannian manifold such that at any point $x \in M^n$ there exists a subspace V_x of $T_x M$ with dimension d such that $K_M(\sigma) \leq 0$ for every plane $\sigma \subset V_x$. If $f : M^n \rightarrow \mathbb{R}^{n+p}$ is an isometric immersion, then $p \geq d$.*

Proof. Since M^n is compact, it is well known that there exist a point $x_0 \in M^n$ and a normal vector $\xi \in N_f M(x_0)$ such that the shape operator A_ξ is positive definite, and in particular $\alpha_{x_0}(X, X) \neq 0$ for every nonzero vector $X \in T_{x_0} M$. Furthermore, $K_{\alpha_{x_0}}(\sigma) \leq 0$ for every plane $\sigma \subset V_{x_0}$ by the Gauss equation. The statement then follows from Corollary 6-(i). \square

For the noncompact case, on the other hand, we have the following immediate consequence of our Corollary 1.

Corollary 7. *Let $f : M^n \rightarrow P^{n+p}$ be an isometric immersion of a complete Riemannian manifold whose radial sectional curvatures satisfy (1.4) into a Hadamard manifold. Assume that at any point $x \in M^n$ there exists a subspace V_x of $T_x M$ with dimension d such that $K_f(\sigma) \leq 0$ for every plane $\sigma \subset V_x$. If $p < d$, then $f(M)$ is unbounded.*

2.2. Omori-Yau maximum principle and Hessian comparison theorem.

The *Omori-Yau maximum principle for the Hessian* is said to hold on a given Riemannian manifold M^n if, for any function $g \in C^2(M)$ with $g^* = \sup_M g < +\infty$, there exists a sequence of points $\{x_k\}_{k \in \mathbb{N}}$ in M^n satisfying:

- (i) $g(x_k) > g^* - \frac{1}{k}$,
- (ii) $\|\text{grad}^M g(x_k)\| < \frac{1}{k}$,
- (iii) $\text{Hess}^M g(x_k)(X, X) \leq \frac{1}{k} \langle X, X \rangle$ for all $X \in T_{x_k} M$.

Such a sequence $\{x_k\}_{k \in \mathbb{N}}$ satisfying (i)-(iii) above is called an *Omori-Yau Hessian sequence* for g . One says that the *Omori-Yau maximum principle* holds if the condition on the Hessian is replaced by a similar one on the Laplacian, namely, if

- (iii) $\Delta^M g(x_k) \leq \frac{1}{k}$.

In this case, $\{x_k\}_{k \in \mathbb{N}}$ is called an *Omori-Yau sequence* for g .

The following is a function theoretic characterization of Riemannian manifolds that satisfy the Omori-Yau maximum principle for the Hessian. For the proof of this, as well as of the other results related to the Omori-Yau maximum principle in this subsection, we refer to [14].

Theorem 8. *Assume that the Riemannian manifold M^n supports a nonnegative function $\gamma \in C^2(M)$ satisfying the following conditions:*

- (a) *γ is proper, that is, $\gamma(x) \rightarrow +\infty$ as $x \rightarrow \infty$,*
- (b) *$\|\text{grad}^M \gamma\| \leq c\sqrt{\gamma}$ outside a compact subset of M^n for some constant $c > 0$,*
- (c) *$\text{Hess}^M \gamma \leq d\sqrt{\gamma F(\sqrt{\gamma})}\langle \cdot, \cdot \rangle$ outside a compact subset of M^n , for some $d > 0$ and some $F \in C^\infty([0, +\infty))$ that satisfies:*
 - (i) $F(0) > 0$, (ii) $F'(t) \geq 0$ on $[0, +\infty)$, (iii) $1/\sqrt{F(t)} \notin L^1[0, +\infty)$.

Then, the Omori-Yau maximum principle for the Hessian holds on M^n .

We point out that, although completeness of M^n is not required in Theorem 8, it is a consequence of the assumptions (a) and (b). Examples of functions satisfying the requirements in Theorem 8 are given by

$$F(t) = A^2 t^2 \prod_{j=1}^J \left(\log^{(j)} t \right)^2, \quad t \gg 1,$$

where $A > 0$ is a constant and $J \geq 1$ is an integer.

Sometimes, for the applications of the maximum principle as in our paper, the following weaker version is enough.

The *Omori-Yau weak maximum principle for the Hessian* is said to hold on a Riemannian manifold M^n if for any function $g \in C^2(M)$ with $g^* = \sup_M g < +\infty$ there exists a sequence of points $\{x_k\}_{k \in \mathbb{N}}$ satisfying:

- (i) $g(x_k) > g^* - \frac{1}{k}$,
- (ii) $\text{Hess}^M g(x_k)(X, X) \leq \frac{1}{k} \langle X, X \rangle$ for all $X \in T_{x_k} M$.

A sequence $\{x_k\}_{k \in \mathbb{N}}$ satisfying (i) and (ii) is called an *Omori-Yau weak Hessian sequence* for g .

Riemannian manifolds that satisfy the Omori-Yau weak maximum principle for the Hessian are characterized as follows.

Theorem 9. *Assume that the Riemannian manifold M^n supports a nonnegative function $\gamma \in C^2(M)$ satisfying the following conditions:*

- (a) *γ is proper,*
- (b) *$\text{Hess}^M \gamma \leq d\sqrt{\gamma F(\sqrt{\gamma})}\langle \cdot, \cdot \rangle$ outside a compact subset of M^n , for some $d > 0$ and some $F \in C^\infty([0, +\infty))$ as in Theorem 8.*

Then, the Omori-Yau weak maximum principle for the Hessian holds on M^n .

Accordingly, it is said that the *Omori-Yau weak maximum principle* holds if (ii) is replaced by the condition

- (ii) $\Delta^M g(x_k) \leq \frac{1}{k}$,

in which case $\{x_k\}_{k \in \mathbb{N}}$ is called an *Omori-Yau weak sequence* for g .

The function theoretic approach to the Omori-Yau Maximum Principle given in Theorem 8 allows one to apply it in different situations, where the choices of γ and F are suggested by the geometric setting. For instance, one has the following special case, where as previously agreed ρ stands for the distance function on a Riemannian manifold M^n to a fixed point.

Theorem 10. *Let M^n be a complete noncompact Riemannian manifold. Assume that $K_M^{\text{rad}} \geq -F(\rho)$, where $F \in C^\infty[0, +\infty)$ satisfies the conditions listed in Theorem 8 and is even at the origin, that is, its derivatives satisfy $F^{(2k+1)}(0) = 0$ for $k \geq 0$. Then, the Omori-Yau maximum principle for the Hessian holds on M^n .*

Remark 3. If we only assume that $\text{Ric}_M(\text{grad}^M \rho) \geq -F(\rho)$, then the conclusion is that the Omori-Yau maximum principle holds on M^n .

The last ingredient for the proof of Theorem 2 is the following version of the well-known Hessian comparison theorem given in [16].

Theorem 11. *Let M^n be a Riemannian manifold and $o, x \in M^n$ be such that there is a minimizing geodesic γ joining o and x , and let ρ be the distance function to o . Suppose that $K_M^{\text{rad}} \leq b$ along γ . If $b > 0$ assume $\rho(x) < \frac{\pi}{2\sqrt{b}}$. Then, we have*

$$\text{Hess}^M \rho \geq C_b(\rho) (\langle \cdot, \cdot \rangle - d\rho \otimes d\rho)$$

along γ .

3. PROOFS

Let $f : M^m \rightarrow \tilde{M}^n$ be an isometric immersion between Riemannian manifolds. Given a function $h \in C^\infty(\tilde{M})$ we set $g = h \circ f \in C^\infty(M)$. Since

$$\langle \text{grad}^M g(x), X \rangle = \langle \text{grad}^{\tilde{M}} h(f(x)), f_* X \rangle$$

for every $x \in M^n$ and $X \in T_x M$, we obtain

$$(3.1) \quad f_* \text{grad}^M g(x) = \left(\text{grad}^{\tilde{M}} h(f(x)) \right)^T,$$

where $(\cdot)^T$ is the tangent component. An easy computation using the Gauss formula gives the well-known relation (see e.g. [6]):

$$(3.2) \quad \text{Hess}^M g(x)(X, Y) = \text{Hess}^{\tilde{M}} h(f(x))(f_* X, f_* Y) + \langle \text{grad}^{\tilde{M}} h(f(x)), \alpha_x(X, Y) \rangle$$

for all $x \in M^n$ and $X, Y \in T_x M$, where α_x stands for the second fundamental form of f at x . In particular, taking traces with respect to an orthonormal frame $\{e_1, \dots, e_m\}$ in $T_x M$ yields

$$\Delta^M g(x) = \sum_{i=1}^m \text{Hess}^{\tilde{M}} h(f(x))(f_* e_i, f_* e_i) + n \langle \text{grad}^{\tilde{M}} h(f(x)), H(x) \rangle,$$

where $H(x) = \frac{1}{n} \sum_{i=1}^m \alpha_x(e_i, e_i)$ is the mean curvature vector of f at x .

Given an isometric immersion $f : M^m \rightarrow N^{n+l} = P^n \times Q^l$, we denote by $\pi_P : N^{n+l} \rightarrow P^n$ and $\pi_Q : N^{n+l} \rightarrow Q^l$ the projections onto P^n and Q^l , respectively. We write (y, z) for points in $N^{n+l} = P^n \times Q^l$ and by abuse of notation denote $y = \pi_P \circ f$ and $z = \pi_Q \circ f$.

Moreover, set

$$\psi_b = \begin{cases} 1 - \cos(\sqrt{b}t) & \text{if } b > 0, \\ t^2 & \text{if } b = 0, \\ \cosh(\sqrt{-b}t) & \text{if } b < 0, \end{cases}$$

where $t > 0$ if $b \leq 0$ and $0 < t < \frac{\pi}{2\sqrt{b}}$ if $b > 0$. Hence $\psi_b'' = C_b \psi_b'$. We define $h \in C^\infty(N)$ by $h = \psi_b \circ r \circ \pi_P$, where r is the distance function on P^n to the reference point o . We call $g = h \circ f$ the *modified radial function* of f .

3.1. Proofs of Theorem 2 and corollaries. As mentioned in Remark 2, Theorem 2 is a consequence of the following stronger result. Here the *algebraic codimension* $p(x)$ of an isometric immersion $f : M^m \rightarrow \tilde{M}^n$ at $x \in M^m$ is the dimension of its first normal space $N_1(x)$ and a sequence of real numbers $\{p_k\}_{k \in \mathbb{N}}$ is said to be *strictly bounded from above* by another $\{q_k\}_{k \in \mathbb{N}}$ if $p_k < q_k$ for all $k \in \mathbb{N}$.

Theorem 12. *Let $f : M^m \rightarrow N^{n+l} = P^n \times Q^l$ be an isometric immersion. Assume that $f(M) \subset B_P[R] \times Q^l$, with $0 < R < \min \left\{ \text{inj}_P(o), \frac{\pi}{2\sqrt{b}} \right\}$, where $\frac{\pi}{2\sqrt{b}}$ is replaced by $+\infty$ if $b \leq 0$. If $K_P^{\text{rad}} \leq b$ in $B_P[R]$, then*

$$(3.3) \quad \liminf \min \left\{ \max_{\sigma \subset W} K_f(\sigma) : \dim f_* W \cap T_{y(x_k)} P > p(x_k) \right\} \geq C_b^2(R)$$

for all Omori-Yau weak Hessian sequence $\{x_k\}_{k \in \mathbb{N}}$ for the modified radial function of f with algebraic codimension sequence $\{p(x_k)\}_{k \in \mathbb{N}}$ strictly bounded from above by $\{\dim f_* T_{x_k} M \cap T_{y(x_k)} P\}_{k \in \mathbb{N}}$. Furthermore,

$$(3.4) \quad \liminf \min \left\{ \max_{\sigma \subset W} K_M(\sigma) : \dim f_* W \cap T_{y(x_k)} P > p(x_k) \right\} \geq C_b^2(R) + \inf_{B_P[R]} K_P.$$

Proof. By the assumption that $f(M) \subset B_P[R] \times Q^l$, the modified radial function g satisfies

$$g^* \leq \psi_b(R) < +\infty,$$

where we write $(\cdot)^* = \sup_M(\cdot)$. Let $\{x_k\}_{k \in \mathbb{N}}$ be an Omori-Yau weak Hessian sequence for g , that is,

- (i) $g(x_k) > g^* - \frac{1}{k}$,
- (ii) $\text{Hess}^M g(x_k) \leq \frac{1}{k} \langle \cdot, \cdot \rangle$.

For each $k \in \mathbb{N}$, take a tangent subspace $W_k \subset T_{x_k} M$ such that $\dim V_k > p_k$, where $V_k = f_*^{-1}(f_* W_k \cap T_{y_k} P)$ and for simplicity of notation we write $p_k = p(x_k)$, $y_k = y(x_k)$. If the algebraic codimension sequence $\{p_k\}_{k \in \mathbb{N}}$ is strictly bounded from above by $\{\dim f_* T_{x_k} M \cap T_{y_k} P\}_{k \in \mathbb{N}}$, then at least $W_k = T_{x_k} M$ satisfies the latter condition, so that the sets on the left-hand side of inequalities (3.3) and (3.4) are nonempty. The idea of the argument is to use (ii) above and (3.2) to estimate $\|\alpha_{x_k}(X, X)\|$ for $X \in V_k$, and then apply Lemma 1 to $\alpha_{x_k}|_{V_k \times V_k}$. This will imply the estimates in the statement. By (3.1), we have

$$\text{grad}^N h(f(x)) = f_* \text{grad}^M g(x) + \left(\text{grad}^N h(f(x)) \right)^\perp,$$

where $(\cdot)^\perp$ denotes the normal component. Note that

$$(3.5) \quad \text{grad}^N h(y, z) = \psi'_b(r(y)) \text{grad}^P r(y).$$

Since h only depends on P^n , we obtain from (3.2) and (3.5) that

$$\begin{aligned} \text{Hess}^M g(x)(X, X) &= \text{Hess}^P \psi_b \circ r(y(x))(X_P, X_P) \\ &\quad + \psi'_b(r(y(x))) \langle \text{grad}^P r(y(x)), \alpha_x(X, X) \rangle, \end{aligned}$$

where $X_P = y_* X$. Observe that

$$\begin{aligned} \text{Hess}^P \psi_b \circ r(y)(X_P, X_P) &= \psi''_b(r(y)) \langle \text{grad}^P r(y), X_P \rangle^2 \\ &\quad + \psi'_b(r(y)) \text{Hess}^P r(y)(X_P, X_P). \end{aligned}$$

Since $\psi_b'' = C_b \psi_b'$, the last two equations yield

$$(3.6) \quad \text{Hess}^M g(x)(X, X) = \psi_b'(r(y(x))) (C_b(r(y(x))) \langle \text{grad}^P r(y(x)), X_P \rangle^2 \\ + \langle \text{grad}^P r(y(x)), \alpha_x(X, X) \rangle + \text{Hess}^P r(y(x))(X_P, X_P)).$$

Theorem 11 gives

$$(3.7) \quad \text{Hess}^P r(y)(Y, Y) = \text{Hess}^P r(y)(Y^\perp, Y^\perp) \\ \geq C_b(r(y)) (\|Y\|^2 - \langle \text{grad}^P r(y), Y \rangle^2),$$

where $Y \in T_y P$ and here Y^\perp is defined by the orthogonal decomposition

$$Y = \langle \text{grad}^P r(y), Y \rangle \text{grad}^P r(y) + Y^\perp.$$

Now, since $X_P = f_* X$ for any $X \in V_k$, we obtain from (3.6) and (3.7) that

$$\text{Hess}^M g(x_k)(X, X) \geq \psi_b'(r_k) \left(C_b(r_k) \|X\|^2 + \langle \text{grad}^P r(y_k), \alpha_{x_k}(X, X) \rangle \right) \\ \geq \psi_b'(r_k) \left(C_b(r_k) \|X\|^2 - \|\alpha_{x_k}(X, X)\| \right),$$

where $r_k = r(y_k)$. Hence, by (ii)

$$\frac{1}{k} \|X\|^2 \geq \psi_b'(r_k) \left(C_b(r_k) \|X\|^2 - \|\alpha_{x_k}(X, X)\| \right)$$

for every x_k and every $X \in V_k$, and therefore,

$$\|\alpha_{x_k}(X, X)\| \geq \left(C_b(r_k) - \frac{1}{k\psi_b'(r_k)} \right) \|X\|^2.$$

Since $g(x_k) = \psi_b(r_k)$ approaches $g^* = \psi_b(r^*) > 0$ by (i) and $\psi_b|_{[0, R]}$ is a homeomorphism onto its image (recall that $R < \frac{\pi}{2\sqrt{b}}$ if $b > 0$), it follows that r_k goes to $r^* > 0$, and in particular $\psi_b'(r_k) \rightarrow \psi_b'(r^*) > 0$. Thus,

$$C_b(r_k) - \frac{1}{k\psi_b'(r_k)} > 0$$

for k sufficiently large and, as $\dim V_k > p_k$, we can apply Lemma 1 to

$$\alpha_{x_k}|_{V_k \times V_k} : V_k \times V_k \rightarrow N_1(x_k).$$

We obtain a plane $\sigma_k \subset V_k$ such that, by the Gauss equation,

$$K_f(\sigma_k) = K_{\alpha_{x_k}}(\sigma_k) \geq \left(C_b(r_k) - \frac{1}{k\psi_b'(r_k)} \right)^2.$$

In particular,

$$\max_{\sigma \subset W_k} K_f(\sigma) \geq \left(C_b(r_k) - \frac{1}{k\psi_b'(r_k)} \right)^2,$$

but since the subspaces $W_k \subset T_{x_k} M$ satisfying $\dim f_*(W_k) \cap T_{y_k} P > p_k$ have been taken arbitrarily, we have indeed

$$\min \left\{ \max_{\sigma \subset W_k} K_f(\sigma) : \dim f_* W_k \cap T_{y_k} P > p_k \right\} \geq \left(C_b(r_k) - \frac{1}{k\psi_b'(r_k)} \right)^2.$$

Then (3.3) follows by letting $k \rightarrow +\infty$. We will now compare the sectional curvatures $K_M(\sigma_k)$ and $K_N(f_* \sigma_k)$. Since $\sigma_k \subset V_k \perp T_{z_k} Q$, $z_k = z(x_k)$, then

$$K_N(f_* \sigma_k) = K_P(y_* \sigma_k).$$

Then, we have that

$$\begin{aligned} K_M(\sigma_k) &= K_f(\sigma_k) + K_P(y_*\sigma_k) \\ &\geq \left(C_b(r_k) - \frac{1}{k\psi'_b(r_k)} \right)^2 + \inf_{B_P[R]} K_P, \end{aligned}$$

and (3.4) follows by a similar argument. \square

Remark 4. That the maximum and minimum on the left-hand side of (3.3) (and similarly for (3.4)) are indeed attained can be argued as follows. At each $x = x_k$, the extrinsic sectional curvature $K_f = K_{\alpha_x} : G_2(T_x M) \rightarrow \mathbb{R}$ is a continuous function on the Grassmannian $G_2(T_x M)$ of (nonoriented) planes in $T_x M$, and by compactness attains its maximum and minimum. Since $G_2(W)$ is a compact subset of $G_2(T_x M)$ for any subspace $W \subset T_x M$, so does the restriction $K_f|_W$. Let

$$\{W_j\}_{j \in \mathbb{N}} \subset \mathcal{W} := \{W \subset T_x M : \dim f_* W \cap T_{y(x)} P > p(x)\} \neq \emptyset$$

be a sequence such that

$$\max_{\sigma \subset W_j} K_f(\sigma) \rightarrow \inf \left\{ \max_{\sigma \subset W} K_f(\sigma) : W \in \mathcal{W} \right\},$$

as $j \rightarrow +\infty$. After passing to a subsequence we can without loss of generality assume that all W_j have the same dimension d and converge to some $W_\infty \in G_d(T_x M)$, where $G_d(T_x M)$ denotes the Grassmannian of (nonoriented) d -planes in $T_x M$. Moreover, since f_* is an isomorphism onto its image, it is clear that the function $W \in G_d(T_x M) \mapsto \dim f_* W \cap T_{y(x)} P$ is upper semicontinuous, and so

$$\dim f_*(W_\infty) \cap T_{y(x)} P > p(x),$$

or equivalently, $W_\infty \in \mathcal{W}$. Hence,

$$\max_{\sigma \subset W_\infty} K_f(\sigma) \in \left\{ \max_{\sigma \subset W} K_f(\sigma) : W \in \mathcal{W} \right\}.$$

Finally, a straightforward contradiction argument allows to conclude that

$$\lim_{j \rightarrow +\infty} \max_{\sigma \subset W_j} K_f(\sigma) = \max_{\sigma \subset W_\infty} K_f(\sigma),$$

and therefore

$$\max_{\sigma \subset W_\infty} K_f(\sigma) = \min \left\{ \max_{\sigma \subset W} K_f(\sigma) : W \in \mathcal{W} \right\}.$$

Proof of Theorem 2. According to Theorem 10, the curvature decay in the statement is sufficient to conclude that the Omori-Yau maximum principle for the Hessian holds on M^m . Thus there exists in M^m an Omori-Yau Hessian sequence $\{x_k\}_{k \in \mathbb{N}}$ for the modified radial function of f , whose algebraic codimension sequence $\{p_k\}_{k \in \mathbb{N}}$ satisfies

$$p_k \leq p < m - l \leq \dim f_* T_{x_k} M \cap T_{y_k} P,$$

that is, $\{p_k\}_{k \in \mathbb{N}}$ is strictly bounded from above by $\{\dim f_* T_{x_k} M \cap T_{y_k} P\}$ as required in Theorem 12. Moreover, given a subspace $W \subset T_{x_k} M$ with $\dim W > p + l$, it is clear that $\dim f_* W \cap T_{y_k} P > p \geq p_k$. In other words,

$$\{W \subset T_{x_k} M : \dim W > p + l\} \subset \{W \subset T_{x_k} M : \dim f_* W \cap T_{y_k} P > p_k\},$$

and in particular

$$\left\{ \max_{\sigma \subset W} K_f(\sigma) : \dim W > p + l \right\} \subset \left\{ \max_{\sigma \subset W} K_f(\sigma) : \dim f_* W \cap T_{y_k} P > p_k \right\}.$$

Therefore,

$$\min \left\{ \max_{\sigma \subset W} K_f(\sigma) : \dim W > p + l \right\} \geq \min \left\{ \max_{\sigma \subset W} K_f(\sigma) : \dim f_* W \cap T_{y_k} P > p_k \right\},$$

and (1.5) follows immediately from (3.3). Similarly, (1.6) follows from (3.4). \square

Proof of Corollary 1. Follows immediately from (1.5) observing that $\inf C_b = \sqrt{-b}$ for $b \leq 0$. \square

Remark 5. If, in addition, $K_P \geq c$ in Corollary 1, then (1.6) provides

$$\sup_M \min \left\{ \max_{\sigma \subset W} K_M(\sigma) : \dim W > p + l \right\} \leq c - b$$

as a criterion of cylindrical unboundness. In particular, when P^{n+p} is either the Euclidean space \mathbb{R}^{n+p} or the hyperbolic space \mathbb{H}^{n+p} and M^n is a complete Riemannian manifold (whose radial sectional curvatures satisfy (1.4)) in which at any point $x \in M^n$ there exists a subspace V_x of $T_x M$ with dimension d such that $K_M(\sigma) \leq 0$ for every plane $\sigma \subset V_x$, we conclude that every isometric immersion $f : M^n \rightarrow P^{n+p}$ with codimension $p < d$ is unbounded (compare with Corollary 7).

Proof of Corollary 3. Here $p = 1$ and $l = 0$, so that (1.6) in Theorem 2 yields

$$\sup_M \min K_M \geq C_b^2(R) + \inf_{B_P[R]} K_P.$$

Since clearly $s_M \geq \min K_M$ at each point, the corollary follows. \square

3.2. Proofs of Theorems 4 and 5.

Proof of Theorem 4. Again consider the modified radial function $g : M^m \rightarrow \mathbb{R}$. Since $\pi_P(f(M)) \subset B_P[R]$, we have that $g^* \leq \psi_b(R)$. Let $\phi : M^m \rightarrow [0, +\infty)$ be given by

$$\phi(x) = \exp \left(\int_0^{|z(x)|} \frac{ds}{\varsigma(s)} \right),$$

where $||$ stands for the distance function to the pole of Q^l . Since f is proper and $\pi_P(f(M)) \subset B_P[R]$, then the function $|z(x)|$ satisfies $|z(x)| \rightarrow +\infty$ as $x \rightarrow \infty$. By hypothesis we have that $\int_0^{+\infty} 1/\varsigma(s) ds = +\infty$ so that $\phi(x) \rightarrow +\infty$ as $x \rightarrow \infty$. We let $x_0 \in M^m$ with $\pi_P(f(x_0)) \neq o$ and set

$$g_k(x) = \frac{g(x) - g(x_0) + 1}{\phi(x)^{1/k}}.$$

Thus $g_k(x_0) > 0$, and since $g^* \leq \psi_b(R) < +\infty$ and $\phi(x) \rightarrow +\infty$ as $x \rightarrow \infty$, we have that $\limsup_{x \rightarrow \infty} g_k(x) \leq 0$. Hence g_k attains a positive absolute maximum at a point $x_k \in M^m$. This procedure yields a sequence $\{x_k\}_{k \in \mathbb{N}}$ such that (passing to a subsequence if necessary) $g(x_k)$ converges to g^* . First suppose that $x_k \rightarrow \infty$

as $k \rightarrow +\infty$. Since g_k attains a maximum at x_k , we have $\text{grad}^M g_k(x_k) = 0$ and $\text{Hess}^M g_k(x_k)(X, X) \leq 0$ for every $X \in T_{x_k}M$. This yields

$$(3.8) \quad \text{grad}^M g(x_k) = \frac{g(x_k) - g(x_0) + 1}{k\phi(x_k)} \text{grad}^M \phi(x_k)$$

and

$$(3.9) \quad \begin{aligned} \text{Hess}^M g(x_k) &\leq \frac{g(x_k) - g(x_0) + 1}{k\phi(x_k)} \left(\text{Hess}^M \phi(x_k) + \left(\frac{1}{k} - 1 \right) \frac{1}{\phi(x_k)} d\phi \otimes d\phi \right) \\ &\leq \frac{g(x_k) - g(x_0) + 1}{k\phi(x_k)} \text{Hess}^M \phi(x_k). \end{aligned}$$

Since $\phi(x) = \zeta(z(x))$, where $\zeta(z) = \exp\left(\int_0^{|z|} ds/\varsigma(s)\right)$, $z \in Q^l$, from (3.2) we have that

$$(3.10) \quad \text{Hess}^M \phi(x)(X, X) = \text{Hess}^Q \zeta(z(x))(X_Q, X_Q) + \langle \text{grad}^Q \zeta(z(x)), \alpha_x(X, X) \rangle$$

for all vectors $X \in T_x M$, where $X_Q = z_* X$. Also observe that

$$\text{grad}^Q \zeta(z) = \frac{\zeta(z)}{\varsigma(|z|)} \text{grad}^Q |z|,$$

and then

$$(3.11) \quad \text{grad}^M \phi(x) = \frac{\phi(x)}{\varsigma(|z(x)|)} \left(\text{grad}^Q |z(x)| \right)^T.$$

Thus, for every $X \in T_x M$ such that $X_Q = 0$, it follows from (3.10) that

$$\begin{aligned} \text{Hess}^M \phi(x)(X, X) &= \frac{\phi(x)}{\varsigma(|z(x)|)} \langle \text{grad}^Q |z(x)|, \alpha_x(X, X) \rangle \\ &\leq \frac{\phi(x)}{\varsigma(|z(x)|)} \|\alpha_x(X, X)\|. \end{aligned}$$

Therefore, by (1.13) we obtain that

$$(3.12) \quad \frac{1}{\phi(x)} \text{Hess}^M \phi(x)(X, X) \leq \frac{\|\alpha_x(X, X)\|}{\varsigma(|z(x)|)} \leq \|X\|^2$$

for every $X \in T_x M$ with $X_Q = 0$. Given $W_{x_k} \subset T_{x_k} M$ with $\dim W_{x_k} > p + l$, we have that the subspace $V_k = f_*^{-1}(f_* W_{x_k} \cap T_{y_k} P)$ has $\dim V_k \geq \dim W_{x_k} - l > p$ and $f_*(V_k)$ is orthogonal to $T_{z_k} Q$. Then, $X_Q = 0$ for every $X \in V_k$, and from (3.9) and (3.12) we get that

$$\begin{aligned} \text{Hess}^M g(x_k)(X, X) &\leq \frac{g(x_k) - g(x_0) + 1}{k\phi(x_k)} \text{Hess}^M \phi(x_k)(X, X) \\ &\leq \frac{\psi_b(R) + 1}{k} \|X\|^2, \end{aligned}$$

for every $X \in V_{x_k}$. Moreover, using Theorem 11, we also have here that

$$(3.13) \quad \text{Hess}^M g(x)(X, X) \geq \psi'_b(r(y(x))) \left(C_b(r(y(x))) \|X\|^2 - \|\alpha_x(X, X)\| \right)$$

for every $X \in V_{x_k}$, since $X_P = X$. Therefore, we obtain that

$$\begin{aligned} \frac{\psi_b(R) + 1}{k} \|X\|^2 &\geq \text{Hess}^M g(x_k)(X, X) \\ &\geq \psi'_b(r_k) \left(C_b(r_k) \|X\|^2 - \|\alpha_{x_k}(X, X)\| \right) \end{aligned}$$

for every x_k and every $X \in V_{x_k}$, where as usual $r_k = r(y_k)$. Hence

$$\|\alpha_{x_k}(X, X)\| \geq \left(C_b(r_k) - \frac{\psi_b(R) + 1}{k\psi'_b(r_k)} \right) \|X\|^2$$

with

$$C_b(r_k) - \frac{\psi_b(R) + 1}{k\psi'_b(r_k)} > 0$$

for k sufficiently large. Reasoning now as in the last part of the proof of Theorem 12, there exists a plane $\sigma_k \subset V_{x_k}$ such that, by the Gauss equation

$$K_f(\sigma_k) = K_\alpha(\sigma_k) \geq \left(C_b(r_k) - \frac{\psi_b(R) + 1}{k\psi'_b(r_k)} \right)^2,$$

and (1.5) and (1.6) follow by letting $k \rightarrow +\infty$ as in the last part of the proof of Theorem 12. To finish the proof of Theorem 4, we need to consider the case where the sequence $\{x_k\}_{k \in \mathbb{N}} \subset M^m$ remains in a compact set. In that case, passing to a subsequence if necessary, we may assume that $x_k \rightarrow x_\infty \in M^m$ and g attains its absolute maximum at x_∞ . Thus $\text{Hess}^M g(x_\infty)(X, X) \leq 0$ for all $X \in T_{x_\infty}M$. In particular, it follows from (3.13) that for every $X \in V_{x_\infty}$

$$0 \geq \text{Hess}^M g(x_\infty)(X, X) \geq \psi'_b(r_\infty) \left(C_b(r_\infty) \|X\|^2 - \|\alpha_{x_\infty}(X, X)\| \right).$$

Therefore

$$\|\alpha_{x_\infty}(X, X)\| \geq C_b(r_\infty) \|X\|^2.$$

By applying Lemma 1 to $\alpha|_{V_{x_\infty} \times V_{x_\infty}} : V_{x_\infty} \times V_{x_\infty} \rightarrow N_f M(x_\infty)$ and reasoning again as in the last part of the proof of Theorem 12, we have that there exists a plane $\sigma_\infty \subset V_{x_\infty}$ such that, by the Gauss equation,

$$K_f(\sigma_\infty) = K_\alpha(\sigma_\infty) \geq C_b^2(r_\infty),$$

and (1.5) follows. Again (1.6) follows as in the last part of the proof of Theorem 12. \square

Proof of Theorem 5. We proceed as in the proof of Theorem 4 to obtain a sequence $\{x_k\}_{k \in \mathbb{N}}$ such that $g(x_k)$ converges to g^* and satisfying

$$(3.14) \quad \text{grad}^M g(x_k) = \frac{g(x_k) - g(x_0) + 1}{k\phi(x_k)} \text{grad}^M \phi(x_k)$$

and

$$(3.15) \quad \text{Hess}^M g(x_k) \leq \frac{g(x_k) - g(x_0) + 1}{k\phi(x_k)} \text{Hess}^M \phi(x_k).$$

Recall that (see (3.11))

$$(3.16) \quad \text{grad}^M \phi(x) = \frac{\phi(x)}{\varsigma(|z(x)|)} \left(\text{grad}^Q |z(x)| \right)^T.$$

Let us first consider the case where $x_k \rightarrow \infty$ as $k \rightarrow +\infty$. From (3.14) and (3.16) we know that

$$\left\| \text{grad}^M g(x_k) \right\| \leq \frac{g^* + 1}{k} \frac{1}{\varsigma(|z_k|)} \leq \frac{\psi_b(R) + 1}{k} \frac{1}{\varsigma(|z_k|)}.$$

Since f is proper and $\pi_P(f(M)) \subset B_P(R)$, then $|z_k| \rightarrow +\infty$ as $k \rightarrow +\infty$. Therefore, taking into account that $\limsup_{t \rightarrow +\infty} 1/\varsigma(t) < +\infty$ we obtain from here that

$$(3.17) \quad \lim_{k \rightarrow +\infty} \left\| \text{grad}^M g(x_k) \right\| = 0.$$

Observe that

$$\text{grad}^N h(f(x)) = \psi'_b(r(y)) \text{grad}^P r(y) = \text{grad}^M g(x) + \left(\text{grad}^N h(f(x)) \right)^\perp,$$

where $y = y(x)$. Therefore,

$$(3.18) \quad \psi'_b(r_k)^2 = \left\| \text{grad}^M g(x_k) \right\|^2 + \left\| \left(\text{grad}^N h(f(x_k)) \right)^\perp \right\|^2,$$

and making $k \rightarrow +\infty$ here we obtain that

$$\lim_{k \rightarrow +\infty} \left\| \left(\text{grad}^N h(f(x_k)) \right)^\perp \right\| = \psi'_b(r^*) > 0,$$

which implies that

$$\left(\text{grad}^N h(f(x_k)) \right)^\perp \neq 0$$

for k sufficiently large. As in the proof of Theorem 4, since $n - l > 1$, given $W_{x_k} \subset T_{x_k}M$ with $\dim W_{x_k} > l + 1$, we have that $V_k = f_*^{-1}(f_*W_{x_k} \cap T_{y_k}P)$ has $\dim V_k \geq \dim W_{x_k} - l > 1$ and $f_*(V_k)$ is orthogonal to $T_{z_k}Q$. Then, using Theorem 11, we also have that

$$(3.19) \quad \text{Hess}^M g(x_k)(X, X) \geq \psi'_b(r_k) \left(C_b(r_k) \|X\|^2 - \|\alpha_{x_k}(X, X)\| \right)$$

for every $X \in V_{x_k}$, since $y_*X = X$. On the other hand, we also know from (3.15) that

$$(3.20) \quad \begin{aligned} \text{Hess}^M g(x_k)(X, X) &\leq \frac{\psi_b(R) + 1}{k} \frac{\text{Hess}^M \phi(x_k)(X, X)}{\phi(x_k)} \\ &= \frac{\psi_b(R) + 1}{k} \frac{1}{\varsigma(|z_k|)} \langle \text{grad}^Q |z_k|, \alpha_{x_k}(X, X) \rangle \end{aligned}$$

for every $X \in T_{x_k}M$. Since we are in codimension one and $\left(\text{grad}^N h(f(x_k)) \right)^\perp \neq 0$ (for k large enough), then

$$(3.21) \quad \alpha_{x_k}(X, X) = \lambda_k(X, X) \left(\text{grad}^N h(f(x_k)) \right)^\perp$$

for a real function λ_k . Now observe that

$$\begin{aligned} \langle \text{grad}^Q |z_k|, \alpha_{x_k}(X, X) \rangle &= \lambda_k(X, X) \langle \text{grad}^Q |z_k|, \left(\text{grad}^N h(f(x_k)) \right)^\perp \rangle \\ &= \lambda_k(X, X) \langle \text{grad}^Q |z_k|, \text{grad}^M g(x_k) \rangle \end{aligned}$$

because of $\langle \text{grad}^Q |z_k|, \text{grad}^P r(y_k) \rangle = 0$. Therefore,

$$\begin{aligned} \langle \text{grad}^Q |z_k|, \alpha_{x_k}(X, X) \rangle &\leq |\lambda_k(X, X)| \left\| \text{grad}^M g(x_k) \right\| \\ &\leq |\lambda_k(X, X)| \frac{\psi_b(R) + 1}{k} \frac{1}{\varsigma(|z_k|)}. \end{aligned}$$

On the other hand, from our hypothesis (1.14) we know that

$$\|\alpha_x(X, X)\| \leq \varsigma^2(|z(x)|) \|X\|^2,$$

and from (3.18) and (3.21) we have that

$$\|\alpha_{x_k}(X, X)\| = |\lambda_k(X, X)| \sqrt{\psi'_b(r_k)^2 - \left\| \text{grad}^M g(x_k) \right\|^2} \leq \varsigma^2(|z_k|) \|X\|^2.$$

That is,

$$\frac{|\lambda_k(X, X)|}{\varsigma(z_k)} \leq \frac{\varsigma(z_k) \|X\|^2}{\sqrt{\psi'_b(r_k)^2 - \left\| \text{grad}^M g(x_k) \right\|^2}}.$$

It follows from here that

$$\langle \text{grad}^Q |z_k|, \alpha_{x_k}(X, X) \rangle \leq \frac{\psi_b(R) + 1}{k} \frac{\varsigma(z_k) \|X\|^2}{\sqrt{\psi'_b(r_k)^2 - \left\| \text{grad}^M g(x_k) \right\|^2}}$$

for every $X \in T_{x_k}M$, so that by (3.20) we get

$$(3.22) \quad \text{Hess}^M g(x_k)(X, X) \leq \left(\frac{\psi_b(R) + 1}{k} \right)^2 \frac{\|X\|^2}{\sqrt{\psi'_b(r_k)^2 - \left\| \text{grad}^M g(x_k) \right\|^2}}.$$

Therefore, from (3.19) and (3.22) we have that

$$\psi'_b(r_k) \left(C_b(r_k) \|X\|^2 - \|\alpha_{x_k}(X, X)\| \right) \leq \left(\frac{\psi_b(R) + 1}{k} \right)^2 \frac{\|X\|^2}{\sqrt{\psi'_b(r_k)^2 - \left\| \text{grad}^M g(x_k) \right\|^2}}$$

for every $X \in V_{x_k}$. Hence

$$\|\alpha_{x_k}(X, X)\| \geq \left(C_b(r_k) - \frac{(\psi_b(R) + 1)^2}{k^2 \psi'_b(r_k) \sqrt{\psi'_b(r_k)^2 - \left\| \text{grad}^M g(x_k) \right\|^2}} \right) \|X\|^2,$$

with

$$\lim_{k \rightarrow +\infty} \left(C_b(r_k) - \frac{(\psi_b(R) + 1)^2}{k^2 \psi'_b(r_k) \sqrt{\psi'_b(r_k)^2 - \left\| \text{grad}^M g(x_k) \right\|^2}} \right) = C_b(r^*) \geq C_b(R) > 0.$$

Reasoning now as in the last part of the proof of Theorem 12, there exists a plane $\sigma_k \subset V_{x_k}$ such that, by the Gauss equation,

$$K_f(\sigma_k) \geq \left(C_b(r_k) - \frac{(\psi_b(R) + 1)^2}{k^2 \psi'_b(r_k) \sqrt{\psi'_b(r_k)^2 - \left\| \text{grad}^M g(x_k) \right\|^2}} \right)^2,$$

and (1.5) and (1.6) follow by letting $k \rightarrow +\infty$ as in the last part of the proof of Theorem 12. Finally, in the case where the sequence $\{x_k\}_{k \in \mathbb{N}} \subset M^n$ remains in a compact subset of M^n , and passing to a subsequence if necessary, we may assume that $x_k \rightarrow x_\infty \in M^n$ and g attains its absolute maximum at x_∞ . Thus, $\text{Hess}^M g(x_\infty)(X, X) \leq 0$ for all $X \in T_{x_\infty} M$. Therefore, it follows again from Theorem 11 that for every $X \in V_{x_\infty}$,

$$0 \geq \text{Hess}^M g(x_\infty)(X, X) \geq \psi'_b(r_\infty) \left(C_b(r_\infty) \|X\|^2 - \|\alpha_{x_\infty}(X, X)\| \right).$$

The proof now finishes as in Theorem 4. \square

4. NOTES

The idea of the proof that any compact surface in \mathbb{R}^3 must have a point of positive Gauss curvature was first taken up by Tompkins [20], who showed that there is no isometric immersion $f : M^n \rightarrow \mathbb{R}^{2n-1}$ if M^n is compact and flat. This result inspired the seminal paper of Chern-Kuiper [4], where Lemma 1 was proved for dimensions $n = 2, 3$ and conjectured to be true for any dimension. This conjecture was proved by Otsuki [13] for $\lambda = 0$ who, consequently, obtained Theorem 6 for all dimensions.

The Chern and Kuiper result gave rise to a long series of works, among others, by O'Neill [12], Stiel [19], Moore [9], Jorge-Koutroufiotis [6], Pigola-Rigoli-Setti [14] and, finally, by Alías-Bessa-Montenegro [3] who obtained Theorem 1 on cylindrically bounded submanifolds.

The maximum principles used throughout this paper, as well as their related results, namely, Theorems 8, 9 and 10, are due to Pigola-Rigoli-Setti [14]. On the other hand, it was shown in [1] that conditions (b) and (c) in Theorem 8 can be replaced by the following equivalent although apparently stronger requirements:

- (b) $\|\text{grad}^M \gamma\| \leq c$ for a constant $c > 0$ outside a compact subset of M^n ,
- (c) $\text{Hess}^M \gamma \leq d \langle \cdot, \cdot \rangle$ for a constant $d > 0$ outside a compact subset of M^n .

A similar observation holds for the Omori-Yau maximum principle.

Regarding complete hypersurfaces of nonpositive Ricci curvature, Leung [8] pioneered their study by proving Theorem 3 in the case $b = 0$ and conjecturing that the assumption on the sectional curvature could be dropped. This, however, turns out not to be true, as shown by Nadirashvili's [10] celebrated counterexample to both Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces. After Leung's work, Smith [17] gave an answer for the case $b < 0$ but with a non-sharp estimate (for having made the Hessian comparison to \mathbb{R}^{n+1} instead of \mathbb{H}_b^{n+1}), and finally Veeravalli [21] obtained Theorem 3.

It is a natural question to ask whether Theorem 2 is still true in the limiting case, that is, when $R = \text{inj}_P(o) = \frac{\pi}{2\sqrt{b}}$, where $\frac{\pi}{2\sqrt{b}}$ is replaced by $+\infty$ if $b \leq 0$. This motivates the following conjecture.

Conjecture 1. *Let $f : M^m \rightarrow N^{n+l} = P^n \times Q^l$ be an isometric immersion with codimension $p = n + l - m < m - l$ of a complete Riemannian manifold. Assume that $R = \text{inj}_P(o) = \frac{\pi}{2\sqrt{b}}$, where $\frac{\pi}{2\sqrt{b}}$ is replaced by $+\infty$ if $b \leq 0$. If $K_P^{\text{rad}} \leq b$ in $B_P[R]$, then*

$$(4.1) \quad \sup_M \min \left\{ \max_{\sigma \subset W} K_f(\sigma) : \dim W > p + l \right\} \geq \max \{-b, 0\}.$$

Moreover,

$$(4.2) \quad \sup_M \min \left\{ \max_{\sigma \subset W} K_M(\sigma) : \dim W > p + l \right\} \geq \max \{-b, 0\} + \inf_{B_P[R]} K_P.$$

It is not clear the extent to which the above conjecture is true, but an affirmative answer at least in the most classic cases, such as $P^n = \mathbb{R}^n$ and $l = 0$, would have deep implications in the field of submanifolds with nonpositive extrinsic curvature. Indeed, Conjecture 1 in this case implies when $p = m - 1$ that a complete Riemannian manifold with sectional curvature $K \leq -c < 0$ cannot be immersed isometrically in \mathbb{R}^{2m-1} , a kind of Efimov's theorem in n dimensions. In particular, this would give us the m -dimensional version of the classical theorem of Hilbert that the hyperbolic plane cannot be realized isometrically in \mathbb{R}^3 .

There is yet another attempt to extend Efimov's theorem to higher dimensions in a different direction proposed independently by Reilly [15] and Yau [22] (see also [23] and Gromov [5]):

“There are no complete hypersurfaces in \mathbb{R}^{n+1} with Ricci curvature $\leq -c$ ” and proved to be very true for $n = 3$ and essentially true for $n > 3$ by Smyth-Xavier [18]. Their main result seems to be inaccessible to techniques using the Omori-Yau maximum principle, and its proof relies on a purely geometric result on the principal curvatures of complete submanifolds of Euclidean space. Still in the case $P^n = \mathbb{R}^n$ and $l = 0$, Conjecture 1 for $p = 1$ would not only settle the above question at all, but also, in the same spirit of Corollary 3, weaken the assumption that $\text{Ric} \leq -c$ to $s \leq -c$.

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